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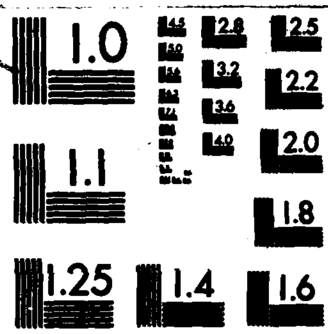

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ON THE LIMITING CONFIGURATION OF INTERFACIAL GRAVITY WAVES

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ABSTRACT

Progressive gravity waves at the interface between two unbounded fluids are considered. The flow in each fluid is taken to be potential flow. The problem is converted into a set of integro-differential equations, reduced to a set of algebraic equations by discretization, and solved by Newton's method together with parameter variation. Meiron and Saffman's calculations showing the existence of overhanging waves are confirmed. However, the present calculations do not support Saffman and Yuen's conjecture that the waves are geometrically limited (i.e. that solutions exist until the interface intersects itself). It is proposed that along a solution branch starting with sinusoidal waves of small amplitude, one reaches solutions with vertical streamlines and then overhanging waves. Continuing on this branch one returns to nonoverhanging waves and thence back toward a wave with vertical streamlines. It is suggested that this succession of patterns and accompanying oscillation in wave characteristics is repeated indefinitely. Graphs of the results are included.

AMS (MOS) Subject Classifications: 76B15, 76C10

Key Words: Interfacial wave, internal wave, potential flow ,

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SIGNIFICANCE AND EXPLANATION

Gravity waves propagating on the interface between two fluids of different densities are observed in laboratory experiments and in oceanographic studies. Internal waves propagating along thermoclines in the ocean and along the interface between salt water and fresh water in estuaries are but two examples. This report is part of an ongoing theoretical and numerical study of this phenomenon.

Here we do a numerical study of the interfacial periodic gravity waves progressing along the interface between two unbounded fluids of different densities. Assuming two-dimensional potential flow in each fluid, the resulting problem is reduced to a set of algebraic equations by discretization and solved by Newton's method together with variation of wave parameters. Calculations of Vanden-Broeck and others are extended to show that progression along a one parameter family of waves toward a limiting configuration is accompanied by oscillatory behavior in the wave characteristics.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON THE LIMITING CONFIGURATION OF
INTERFACIAL GRAVITY WAVES

R.E.L. Turner and J.-M Vanden-Broeck

1. Introduction

We consider two-dimensional progressive gravity waves propagating at the interface between two unbounded fluids of different densities ρ_1 and ρ_2 where ρ_1 is the density of the lower fluid. For $\rho_2 = 0$ the problem reduces to that of the classical Stokes' waves. Stokes proposed that these waves reach their limiting configuration when the maximum velocity in the horizontal direction equals the phase speed of the wave. In a frame of reference moving with the wave, the velocity at the crest is then equal to zero and the free surface makes a 120° angle with itself. Stokes' conjecture has been confirmed in the work of Amick, Fraenkel, and Toland[1]. For $\rho_2 > 0$, Stokes' limiting configuration is no longer possible because such a stagnation point in the lower fluid would result in an infinite velocity in the upper fluid at that point.

Conjectures about the limiting configuration of interfacial waves were made by Holyer[2] and Saffman and Yuen[3]. Holyer[2] performed extensive computations by using Padé approximants and found waves for which the free surface profile is vertical at some point. In a similar problem with the fluids confined between horizontal walls it is shown analytically by Amick and Turner[4] that if the solitary waves do not become infinitely broad in the horizontal direction then vertical streamlines must appear. Based on her calculations Holyer[2] conjectured that a wave with a vertical profile at some point constituted a limiting configuration. Since the horizontal velocity at such a point would equal the phase speed, Holyer's[2] criterion reduces to Stokes' criterion when $\rho_2 = 0$. Saffman and Yuen[3] pointed out that there are no dynamical or kinematical reasons to reject interfacial waves for which the horizontal velocity exceeds the phase velocity at some

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point. The existence of such waves was subsequently demonstrated numerically by Meiron and Saffman[5]. For these waves portions of the heavier fluid lie above lighter fluid and for this reason Meiron and Saffman[5] called them "overhanging" waves (cf figure 3c).

The computations of Meiron and Saffman[5] have thus shown that the appearance of a vertical tangent on the interface does not correspond to a limiting configuration. Saffman and Yuen[3] proposed that the overhanging waves continue to exist until the interface intersects itself. However, the calculations of Meiron and Saffman[5] were stopped well before any such limiting configuration and thus do not confirm the conjecture of Saffman and Yuen[3].

In the present paper we compute interfacial waves by using Vanden-Broeck's[6] scheme (hereafter referred to as V-B). The problem is first formulated as an integro-differential set of equations. These equations are then reduced to a set of algebraic equations through discretization and the resulting equations solved by Newton's method. Our results confirm the existence of overhanging waves as computed by Meiron and Saffman[5]. Moreover, our scheme enabled us to extend their calculations. Continuing along the branch of solutions we compute, we find that there is a return from the overhanging configuration to one in which the fluid interface is a single-valued function of the horizontal variable. We believe that further progress along the branch will produce steepening of the streamline and development of a second overhanging wave. Accompanying this oscillation in shape is an oscillation in speed and amplitude as clearly shown in Fig. 2. We conjecture that an alternation between nonoverhanging and overhanging waves continues indefinitely as one proceeds along the solution branch. Such oscillatory behavior has been exhibited in the analytical work of Longuet-Higgins and Fox[7] on the surface wave problem.

2. Formulation

Let us consider waves at the interface between two fluids of infinite extent having densities ρ_1 and ρ_2, where ρ_1 is the density of the lower fluid. Assume a wave of wavelength λ propagates with phase velocity c under the influence of gravity g. The variables are made dimensionless by taking λ as the unit length and c as the unit velocity. We choose a Cartesian frame of reference in which the flow is steady. The x-axis is parallel to the velocity at an infinite distance from the interface and the y-axis is directed vertically upward so that gravity acts in the negative y direction. It is assumed that the interface is symmetric with respect to the y axis.

Let ϕ_1 and ϕ_2 be potential functions and let ψ_1 and ψ_2 be stream functions in the lower and upper fluids respectively. Without loss of generality we choose $\phi_1 = \phi_2 = 0$ at a crest and $\psi_1 = \psi_2 = 0$ at the interface. The wave speed parameter, the density parameter, and the steepness are defined by the relations

$$\mu = \frac{2\pi c^2}{g\lambda} \tag{1}$$

$$\rho = \frac{\rho_2}{\rho_1} \tag{2}$$

$$s = y(0) - y\left(\tfrac{1}{2}\right) \tag{3}$$

By choosing an origin for y so that the difference in the total Bernoulli head from the lower fluids to the upper is $(\rho_2 - \rho_1)\frac{c^2}{2}$, the condition that the pressure be continuous at the interface is expresed by

$$\frac{\mu}{4\pi}\left(q_1^2 - \rho q_2^2\right) + (1 - \rho)y = \frac{\mu}{4\pi}(1 - \rho) \tag{4}$$

Here q_1 and q_2 are the magnitudes of the dimensionless velocities where $\psi_1 = 0$ and $\psi_2 = 0$. Note that the choice of the origin for y is such that in an undisturbed fluid the interface is at $y = 0$.

Now consider x and y as functions of ϕ_1 and ψ_2 in the lower fluid and as functions of ϕ_2 and ψ_2 in the upper fluid. This provides two different parametric representations for the interface; namely, $x(\phi_1, 0-)$, $y(\phi_1, 0-)$ and $x(\phi_2, 0+)$, $y(\phi_2, 0+)$ where $0-$ and $0+$ indicate that the second variable has been allowed to approch zero from below and above, respectively. We use the notation $x_1(\phi_1)$, $y_1(\phi_1)$ and $x_2(\phi_2)$, $y_2(\phi_2)$, respectively, for these boundary values. In the present context the Hilbert transform provides the integral relations

$$\frac{dx_1}{d\phi_1} = 1 - \int_0^{1/2} \frac{dy_1}{d\phi_1'} \left[\cot \pi(\phi_1' - \phi_1) - \cot \pi(\phi_1' + \phi_1)\right] d\phi_1' \tag{5}$$

and

$$\frac{dx_2}{d\phi_2} = 1 + \int_0^{1/2} \frac{dy_2}{d\phi_2'} \left[\cot \pi(\phi_2' - \phi_2) - \cot \pi(\phi_2' + \phi_2)\right] d\phi_2' \tag{6}$$

(cf V-B[6]).

In general, $\phi_1 \neq \phi_2$ for two points in contact on the interface. To relate the values of the potentials we define the function

$$\phi_1 = g(\phi_2) \tag{7}$$

by the "contact relations"

$$x_2(\phi_2) = x_1\big(g(\phi_2)\big);\ y_2(\phi_2) = y_1\big(g(\phi_2)\big) \tag{8}$$

and normalize g by requiring $g(0) = 0$. In the paper V-B the basic equations were written using ϕ_1 as the independent variable and a change of variables was used to have mesh points equally spaced in ϕ_2, thereby concentrating points where the velocity in the upper fluid changes rapidly. Here the same end is accomplished by using ϕ_2 as the basic independent variable. By differentiating the relations in (8) one can rewrite equation (4) as

$$\frac{\mu}{4\pi}\left[\left(\frac{dg}{d\phi_2}\right)^2 - \rho\right]\left[\left(\frac{dx_2}{d\phi_2}\right)^2 + \left(\frac{dy_2}{d\phi_2}\right)^2\right]^{-1} \times (1-\rho) y_2(\phi_2) = \frac{\mu}{4\pi}(1-\rho) \tag{9}$$

and equation (5) as

$$\frac{dx_2}{d\phi_2}\left(\frac{dg}{d\phi_2}\right)^{-1} = 1 - \int_0^{1/2} \frac{dy_2}{d\phi_2'} \left\{\cot \pi\left[g(\phi_2') - g(\phi_2)\right] - \cot \pi\left[g(\phi_2') + g(\phi_2)\right] d\phi_2'\right. \tag{10}$$

The problem can now be formulated as follows: fix two of the three parameters ρ, μ, and s; then find functions $x_2(\phi_2)$, $y_2(\phi_2)$, and $g(\phi_2)$ defined for $\phi_2 \in [0, \frac{1}{2}]$, and a value for the remaining parameter so that (6), (9), and (10) are simultaneously satisfied. We solve the problem numerically. The relations (6), (9) and (10) are replaced by algebraic equations to be satisfied at discrete, equally spaced values of the variable ϕ_2 and the resulting algebraic equations are solved using Newton's method. For more details regarding the numerical procedure see V-B. In that paper ρ and s were given and μ was found as part of the solution. To obtain the results presented here we used the scheme from V-B as well as a scheme with the additional option of fixing ρ and μ and finding s as part of the solution.

3. Numerical results

The scheme outlined in the previous section was used in V-B to compute interfacial waves for $\rho = 0.1$ with $N < 25$, where N denotes the number of mesh points between $\phi_2 = 0$ and $\phi_2 = \frac{1}{2}$. As can be seen from table 1, the results in V-B[6] agree with those of Holyer[2]. Saffman and Yuen[3] repeated the calculations by using a numerical scheme based on series truncation and table 1 shows agreement to five decimal places with the results of Holyer[2] and V-B[6] providing a valuable check on all three procedures.

s	Holyer	Saffman & Yuen	Vanden-Broeck (N = 25)	N = 30	N = 60
$\frac{.6}{2\pi}$	.880628	.880626	.880624	.880625	.880626
$\frac{.8}{2\pi}$	.930496	.930484	.930484	.930484	.930484

Table 1. Selected values of the speed parameter μ as a function of steepness s for $\rho = 0.1$ from Saffman and Yuen[3], Vanden-Broeck[6], and present calculations. Saffman and Yuen[3] use $H = 2\pi s$ and $[(1+\rho)\mu/(1-\rho)]^{1/2}$ for steepness and speed, respectively.

In V-B solutions were obtained starting from small amplitude waves and continuing up to a wave with a profile which is almost vertical at some distance from the crest (cf. V-B, figure 1). Overhanging waves were not obtained in V-B due to the use of relatively few mesh points. The computations from V-B[6] are continued in the present work, wherein more mesh points are used (N = 30, 45, 60) and continuation along a branch of solutions is achieved by switching between μ and s for parameterization. The convergence with increasing values of N can be seen from the selected valves in table 1.

Our results with $N = 60$ agree with those of Meiron and Saffman[5] and confirm the existence of overhanging waves. Moreover, our scheme enabled us to carry forward Meiron and Saffman's[5] calculations. Fig. 1 shows the solution pairs (s,μ) for ρ taking the values 0.1, 0.5, and 0.7. The broken line (in Fig. 2) corresponds to the results obtained by Meiron and Saffman[5] for $\rho = 0.1$ while the solid curve shows our extension of their results. Fig. 2, for $\rho = 0.1$, shows in some detail the oscillation referred to in the introduction and will be discussed more fully in the following paragraphs.

Apart from the shift in the solution set with changing ρ, which can be seen in Fig. 1, the characteristics of the waves at various points along each solution branch were substantially unaltered by changes in ρ. To envision the variation in wave characteristics for a fixed ρ it is useful to envision moving on a solution branch using arclength as a parameter. We describe the behavior for the case $\rho = 0.1$ which appears typical and for which we have the most extensive data.

Starting with an infinitesimal steepness s and a value of $\mu = \frac{(1 - \rho)}{(1 + \rho)} = .818$ progression along the branch initially produces an increase in both s and μ (cf figure 1, $\rho = 0.1$) a typical wave profile along this section is shown in figure 3a, the letter "a" corresponding to the point "a" in figure 2. The increase in s and μ on this first section is accompanied by an increase in the maximum slope of the interface up to the point "b" at which the interface has a vertical tangent. The corresponding wave is shown in figure 3b. Continuing past "b" one reaches a maximum for s and shortly thereafter a maximum for μ. Past point "b" and up to the point "e" the wave profile is no longer a graph over x, but is "overhanging" as pictured in figure 3c corresponding to point "c" where there appears to be the greatest overhang. At point "d" there is a slight overhang and at point "e" it appears to have disappeared so that the profile at "e" resembles that at "b". In progressing past point "e" one sees further evidence of oscillation in s and μ. The computed profiles past "e" had no overhang and resembled the wave in figure 3a. Past the last point shown in figure 2 the computations became unreliable and were discontinued.

Conclusion

The results of Longuet-Higgins and Fox[7] show that the speed of surface waves (corresponding to $\rho = 0$) oscillates infinitely often as the wave of greatest height is approached. Here we have given numerical evidence that a similar oscillation occurs for interfacial waves with density ratio $\rho > 0$. Whereas the steepness did not oscillate in the calculations of Longuet-Higgins and Fox[7], we find evidence of oscillation in both the speed and steepness parmeters in the case that $\rho > 0$. The accompanying profiles of waves which go from nonoverhanging to overhanging and thence back to the nonoverhanging configuration lead us to conjecture that the limiting behavior is characterized by infinitely many oscillations in all of the characteristics of the waves.

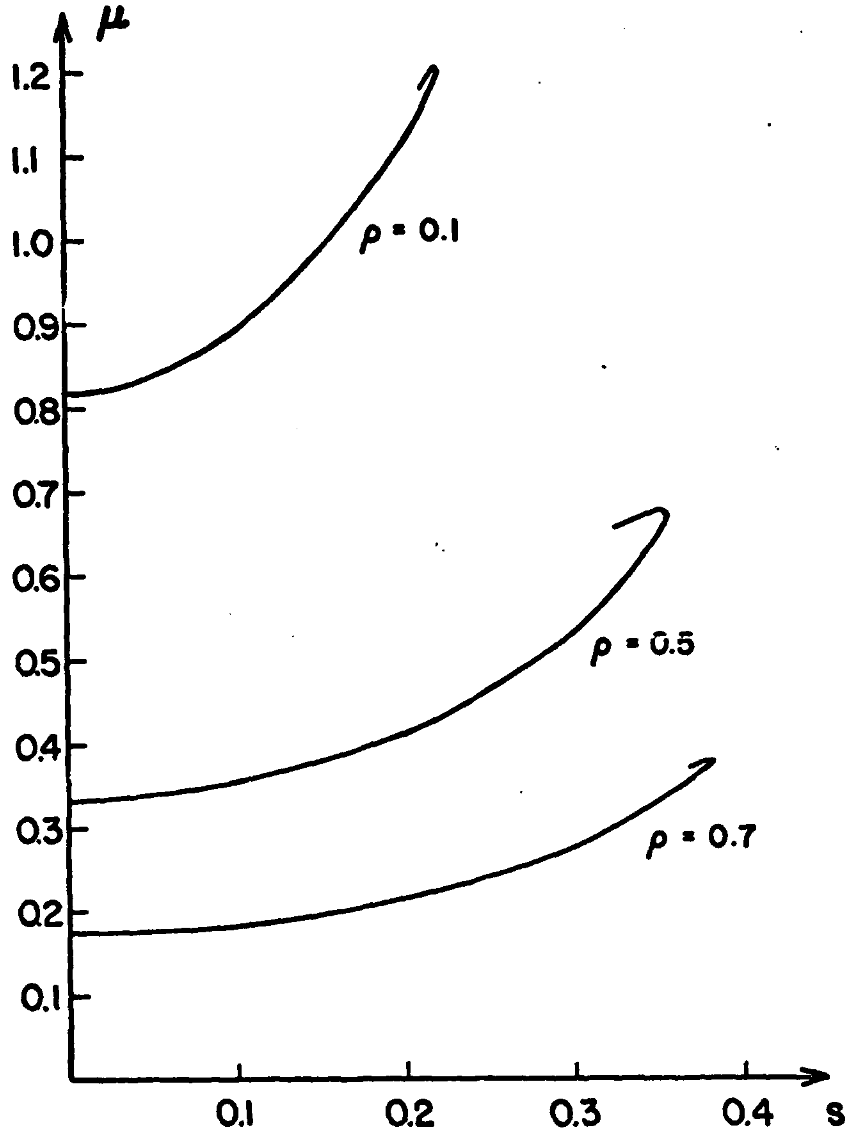


Figure 1. Values of the speed parameter μ versus the steepness s for $\rho = 0.1$, 0.5 and 0.7.

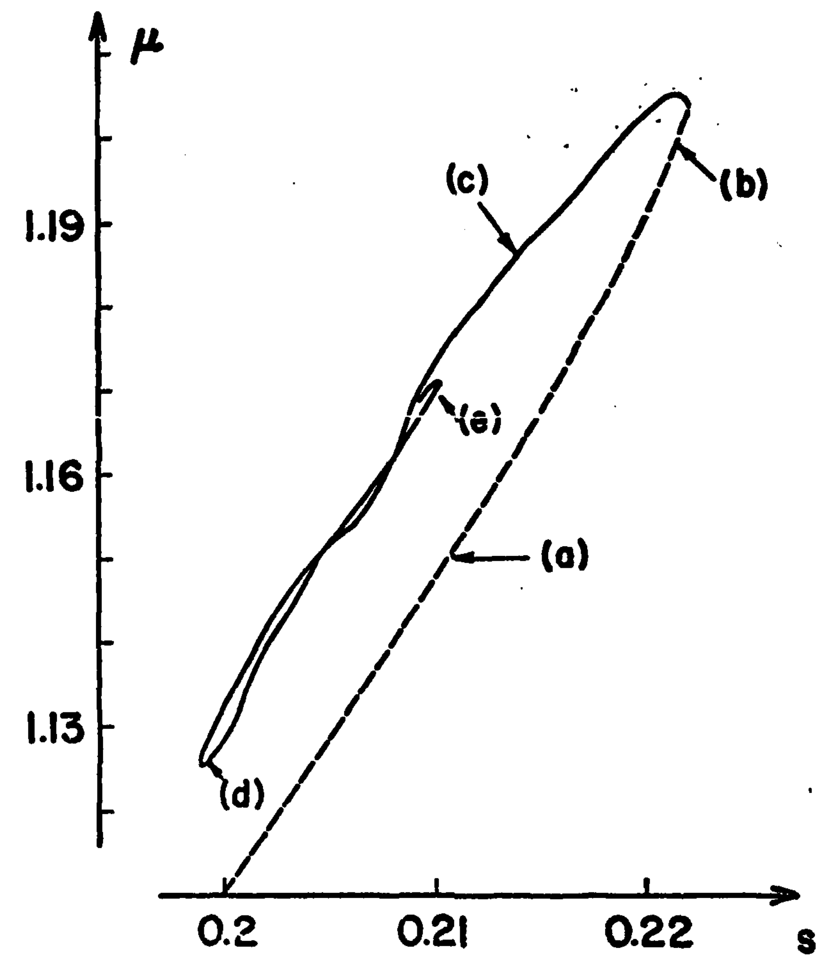


Figure 2. Values of the speed parameter μ versus the steepness s for $\rho = 0.1$.

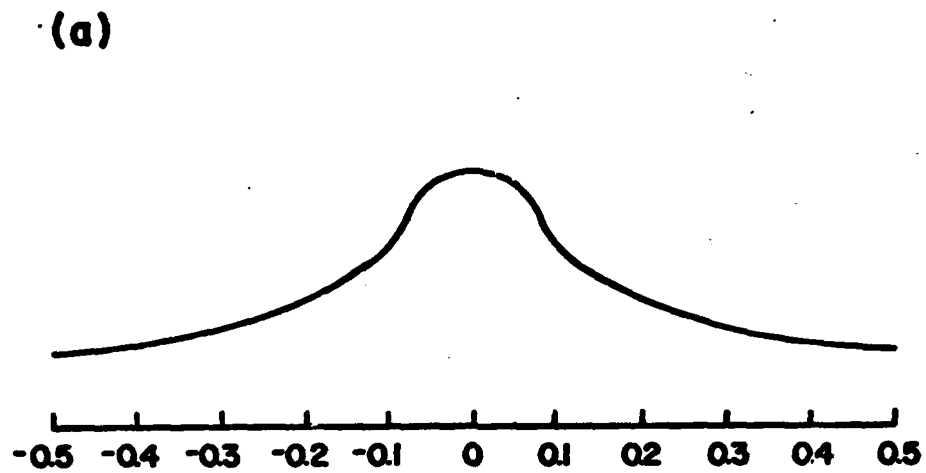
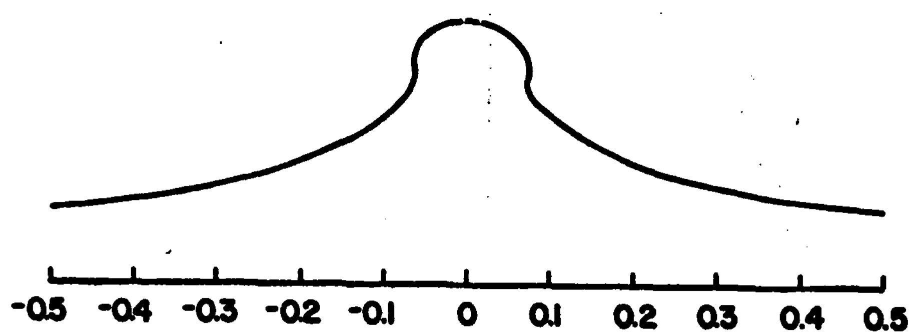
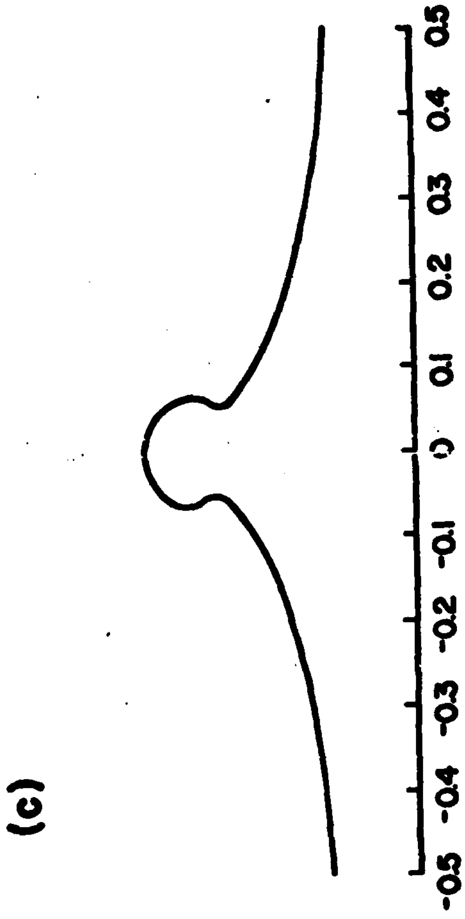
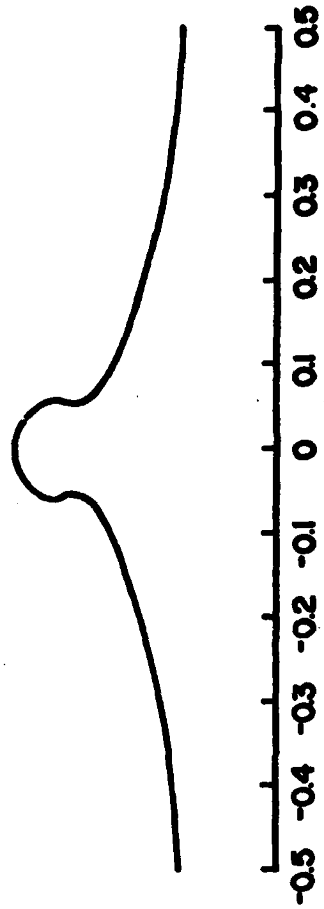


Figure 3. Computed profiles of the waves for $\rho = 0.1$ and various values of μ and s. The vertical scale is the same as the horizontal scale. The letters (a,), (b), (c), (d), and (e) correspond to the points a, b, c, d, and e in Figure 2.

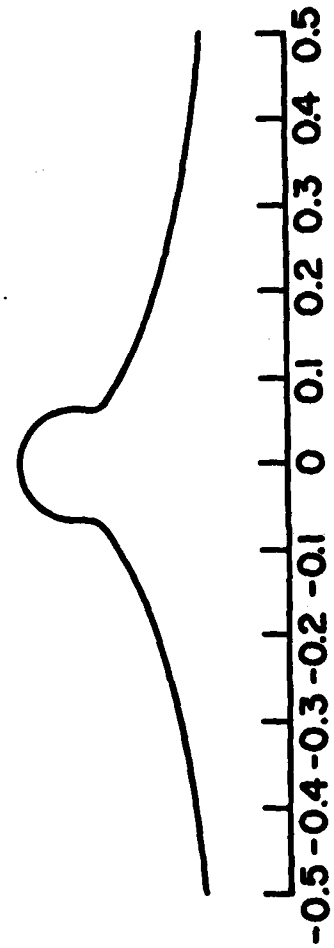
(b)




(c)
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5



(d)



(e)

REFERENCES

[1]C.J. Amick, L.E. Fraenkel, and J.F. Toland, Acta Math., 148, 193 (1982)

[2]J. Y. Holyer, J. Fluid Mech. 93, 433 (1979)

[3]P. G. Saffman and H. C. Yuen, J. Fluid Mech. 123, 459 (1982)

[4]C.J. Amick and R.E.L. Turner, to appear

[5]D. I. Meiron and P. G. Saffman, J. Fluid Mech. 129, 213 (1983)

[6]J. M. Vanden-Broeck, Phys. Fluids 23, 1723 (1980)

[7]M. S. Longuet-Higgins and M.J.H. Fox, J. Fluid Mech. 85, 769 (1978)

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ABSTRACT (cont.)

20. limited (i.e. that solutions exist until the interface intersects itself). It is proposed that along a solution branch starting with sinusoidal waves of small amplitude, one reaches solutions with vertical streamlines and then overhanging waves. Continuing on this branch one returns to nonoverhanging waves and thence back toward a wave with vertical streamlines. It is suggested that this succession of patterns and accompanying oscillation in wave characteristics is repeated indefinitely. Graphs of the results are included.

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